Chapter 4

How Truth-Taking and Truth-Making Share a Rational Form

In the first chapter, we have laid out the basics of an account of open reason relations. We pursued a pragmatics-first explanatory strategy, which starts with the practice of giving reasons for and against claims. This practice is constrained by reason relations between contents of discursive acts. Something is a reason for something else only if the second is implied by the first. Something is a reason against something else only if the second is incompatible with the first. Being inferentialists, we take these reason relations to be constitutive of the contents. Being logical expressivists, we claim that it is the characteristic function of logical vocabulary to allow us to express, in the form of assertions and denials, what stands to what in these reason relations of implication and incompatibility. In this way logical vocabulary allows us to make explicit the contents of our expressions.

In the second and third chapter, we showed how we can extend open reason relations among atomic sentences to logically complex sentences. In particular, we can encode both kinds of reason relations among atomic sentences, implication and incompatibility, in a single consequence relation over these sentences. We can then add logical connectives to our language and close the consequence relation under suitable sequent rules. If the consequence relation over the atomic sentences is nonmontonic, then the resulting consequence relation over the whole language is nonmonotonic and nontransitive. It includes, however, every classically valid inference (in the multiple-conclusion framework). We can introduce object language operators that make explicit which inferences are monotonic and which are classically valid. The formalism also allows us to spell out in more detail what it means for logical vocabulary to make explicit what implies what and what is incompatible with what.

We have thus taken logic to be concerned with the norms that govern discourse. And this is true in two ways: First, we introduced logical vocabulary by specifying its role in reason relations (via giving sequent rules), which constrain the norms of discourse. Second, we claimed that it is the characteristic job of logical vocabulary to make explicit the reason relations, and hence constraints on discursive norms for non-logical vocabulary.

In this chapter, we will explore how this relates to an approach that takes logic to be concerned with worldly states of affairs, namely a variant of Kit Fine's (2017a; 2017b) exact truth-maker theory. We will thus compare the pragmatics-first approach that we pursued so far with a semantics-first approach that starts with the idea that what it is for something to have content is to represent reality to be a particular way.[1]

It will turn out that the approach that starts with norms of discourse and the approach that starts with worldly states are isomorphic, in the ideal case in which our conceptual norms match the structure of alethic modality. Both can encode the same consequence relations, and they do so

[1]The ideas in this chapter have been developed in conjunction with the ideas and results presented in (Hlobil, 2022a). That paper doesn't, however, include any treatment of nonmonotonic consequence relations.

in structurally identical ways. More specifically, the pragmatic-normative relations among discursive acts share their structure with the alethic-modal relations among the worldly states that they represent. Thus, we find the same abstract structure twice, once on the side of discursive norms—the structure of truth-taking—and a second time on the side of the possibility and impossibility of states—the structure of truth-making.

Discursive acts and worldly states play corresponding roles relative to other discursive acts and worldly states, respectively. We will call the role of an item in a structure its "form." If two structures are identical up to isomorphism, then the corresponding items share their form (their *morphe* is *isos*). Hence, discursive acts and worldly states have the same forms, relative to their respective pragmatic-normative and alethic-modal structures. We call these forms their "rational forms." Since these structures are isomorphic and discursive acts map onto the states that they represent under this isomorphism, a discursive act and the state it represents share their rational form. The rational form is what our discourse about a part of reality and that part of reality have in common.

The isomorphism between the two frameworks thus allows us to say that what we grasp in grasping a content is the rational form that our discursive act and that part of reality that it is about have in common. Thus, content manifests itself in two ways: in the alethic-modal relations among worldly states and in the pragmatic-normative relations among discursive acts. These two manifestations of content share their form but differ in their matter. We hence have a bi-modal, hylomorphic conception of content. In this chapter, we will start to articulate this conception of content by explaining the isomorphism between the account from the previous chapters and truth-maker theory.

4.1 Truth-Maker Theory

The goal of this section is to put in place a version of truth-maker theory, which we will later show to be isomorphic to the theory that we put forward in the previous chapters. The ability of the sequent calculi in the previous chapter to codify open reason relations will, hence, extend to this version of truth-maker theory. Thus, the truth-maker theory below will be substructural in the same way in which the sequent calculus in the previous chapter was substructural. In order to meet our desideratum of being able to codify open reason relations, we will start with a structural version of truth-maker theory and then show how we can drop the constraints that prevent the structural version from codifying open reason relations.

We can think of this variant of truth-maker theory as a representational approach to content. The content of a sentence—or a speech act or belief—is explained in terms of the worldly states it represents. Thus, truth-maker theory pursues the opposite order of explanation of the one we pursued in the previous two chapters. That makes it particularly interesting to see the common structure in these two approaches. So let's start by spelling out the variant of truth-maker theory that we are going to use.

According to truth-maker theory, there are parts or aspects of reality, called "states," that make some sentences true, i.e. verify them, and make some sentences false, i.e. falsify them (Fine, 2017a, 2014). States can be parts of other states. Some states are possible and others are impossible. Let's follow Fine (2017a, 647) and say that a "modalized state space" is a triple of a set of states, the subset of those states that are possible, and the parthood relation among these states:

Definition 1. *Modalized state space*: A modalized state space is a triple, $\langle S, S^{\Diamond}, \sqsubseteq \rangle$, such that S is a non-empty set of states, $S^{\Diamond} \subseteq S$ (intuitively: the possible states), and $\sqsubseteq$ is a partial order on S (intuitively: parthood), such that all subsets of S have a least upper bound.

States can be combined—or, as we shall say, "fused"—to yield further states. We define the fusion of states as the smallest state of which all the initial states are parts, i.e., as the least upper bound of the initial set of states under our partial order (Fine, 2017a, 646).

Definition 2. *Fusion*: The fusion of a set of states $T = \{t_1, t_2, t_3, ...\}$, written $t_1 \sqcup t_2 \sqcup t_3...$ or $\bigsqcup T$, is the least upper bound of T with respect to $\sqsubseteq$.

Every modalized state space has a least element, $\square$, which is the "null state" that is part of every state. It is the least upper bound of the empty set.

According to truth-maker theory, the meaning of a sentence is a pair, $\langle V, F\rangle$, of the set of its verifiers, V, and the set of its falsifiers, F. We call such meanings "bilateral propositions." The verifying states (truth-makers) and falsifying states (falsity-makers) of a sentence are wholly relevant to the sentence. The state of it raining, e.g., is a truth-maker of the sentence "It is raining." But the state of it raining and it being cold is not a truth-maker of "It is raining" because it is not wholly relevant to the truth of the sentence (its being cold is irrelevant).

Given a language $\mathfrak{L}$, we can assign every sentence its meaning, i.e., a bilateral proposition. The interpretation of a sentence A, written $|A|$, is hence the pair consisting of the set of states that make A true, written $|A|^+$, and the set of states that make A false, written $|A|^-$. A model is a modalized state space together with an interpretation function.

Definition 3. *Model*: Given a language $\mathfrak{L}$, a model, $\mathcal{M}$, is a quadruple $\langle S, S^\lozenge, \sqsubseteq, |\cdot|\rangle$, where $\langle S, S^\lozenge, \sqsubseteq\rangle$ is a modalized state space and $|\cdot|$ is a interpretation function, such that $|A| = \left\langle |A|^+, |A|^- \right\rangle \in \mathcal{P}(S) \times \mathcal{P}(S)$.

We write $\mathcal{M}, s \Vdash A$ if a state s verifies a sentence A in model $\mathcal{M}$, and if no risk of confusion arises, simply $s \Vdash A$. Similarly, we write $\mathcal{M}, s \dashv\mid A$ to say that s is a falsifier of sentence A in model $\mathcal{M}$. In other words, relative to a given model, $|A|^+$ is the same as $\{s : s \Vdash A\}$, and $|A|^-$ is the same as $\{s : s \dashV A\}$.

If we want to use these models to do logic, we have to treat some bits of vocabulary as logical vocabulary by keeping its meaning fixed in all models. So let $\mathfrak{L}$ be a language that results from adding $\neg$ and $\wedge$ and $\vee$ to a stock of atomic sentences $\mathfrak{L}_0$. To hold the meanings of logical vocabulary fixed, we give semantic clauses that specify the truth-makers and falsity-makers of complex sentences in terms of the truth-makers and falsity-makers of their parts.[2] We will limit ourselves throughout to propositional logic, as we did in the previous chapter.[3] For the atomic sentences, our models directly specify their truth-makers and falsity-makers. Thus, we have:

(atom+) $s \Vdash p$ iff $s \in |p|^+$

(atom−) $s \dashV p$ iff $s \in |p|^-$

For conjunction, it is plausible that the truth-makers of conjunctions are fusions of the truth-makers of their conjuncts. For example, the sentence "It is raining and it is night" is made true by situations that combine a truth-maker of "It is raining" and a truth-maker of "It is night" (and nothing else). And it also seems plausible that a conjunction is made false by

[2]Thus the meanings of logical connectives are functions from some (bilateral) propositions to a (bilateral) proposition.

[3]We cannot foresee any particular problems with the extension of our results to first-order logic. However, there are well-known unresolved questions regarding the truth- and falsity-makers of universal generalizations (Fine 2017c, sec I.7; Jago 2018, chap 5). For example, the truth-maker of "All humans are mammals" cannot just be a fusion of the truth-makers of its actual instances because this fusion doesn't rule out the possibility that there is another human who is not a mammal. One option is to add totality-facts and to say that a truth-maker of a universal generalization is a fusion of truth-makers for all its instances and the totality-fact that these are all the instances (Armstrong, 2004, 19). Fine prefers a solution in terms of arbitrary objects (Fine, 2017c, 568-569). We suspect that any plausible treatment of these issues can be reproduced within normative bilateralism, so that the isomorphism that is our topic here will continue to hold for the resulting first-order systems.

any state that makes one of the conjuncts false. For example, the state of it being sunny and the state of it being daytime are intuitively both falsity-makers for "It is raining and it is night." Thus, we have the following semantic clauses for conjunction:

(and+) $s \Vdash B \wedge C$ iff $\exists u, t\, (u \Vdash B$ and $t \Vdash C$ and $s = u \sqcup t)$

(and−) $s \dashV B \wedge C$ iff $s \dashV B$ or $s \dashV C$

Disjunction works, plausibly, in an analogous way, with truth-making flipped to falsity-making and vice versa: A state makes a disjunction true if it makes one of the disjuncts true. And it makes the disjunction false if it makes both disjuncts false.

(or+) $s \Vdash B \vee C$ iff $s \Vdash B$ or $s \Vdash C$

(or−) $s \dashV B \vee C$ iff $\exists u, t\, (u \dashV B$ and $t \dashV C$ and $s = u \sqcup t)$

And for negation, it is plausible to assume that a state makes a negation true if it makes the negatum false, and it makes a negation false if it makes the negatum true.

(neg+) $s \Vdash \neg B$ iff $s \dashV B$

(neg−) $s \dashV \neg B$ iff $s \Vdash B$

It is a tricky question what the falsity makers of many sentences are, and hence what the truth-makers for their negations are. What is, e.g., the falsity-maker of "Wittgenstein was French" and, hence, the truth-maker of "Wittgenstein was not French"? Is it the state of him having been Austrian? Or do we have to posit a negative state of Wittgenstein failing to be French? Or is it the state of the totality of the states that make up actuality not containing a truth-maker for "Wittgenstein was French"? We will leave such questions unanswered. We will simply assume that we have suitable falsity makers for our atomic sentences. We allow ourselves this liberty because we will ultimately not endorse the semantics-first order of explanation that would make these questions very pressing. Rather, we will suggest below that we understand falsity-makers in terms of the speech act of denying. But we first need to get truth-maker theory on the table.

It will prove convenient below to not only talk of individual sentences being made true or false but also of sets of sentences being made true or false. We will say that a set of sentences is made true by the truth-makers of the conjunctions of its members, writing $\bigwedge\{\gamma_1, ..., \gamma_k\}$ for $\gamma_1 \wedge ... \wedge \gamma_k$, unless there are no such truth-makers, in which case it is made true by the null state. Similarly, we say that a set of sentences is made false by the falsity-makers of the disjunction of its members, unless there are no such falsity-makers.

Definition 4. *Truth- and falsity-makers of sets*: $u \Vdash \Gamma$ iff $u \Vdash \bigwedge \Gamma$, unless $\{x : x \Vdash \bigwedge \Gamma\} = \emptyset$ in which case $\square$ and nothing else makes Γ true. And $t \dashv\Vert \Delta$ iff $t \dashv\Vert \bigvee \Delta$, unless $\{x : x \dashv\Vert \bigvee \Delta\} = \emptyset$ in which case $\square$ and nothing else makes Δ false.

Given the clauses for conjunction and disjunction, it is easy to see that the truth-makers of sets are the fusions of the truth-makers of their members, if there are any. And the falsity-makers of sets are the fusions of the falsity-makers of their members, if there are any. So the appeal to conjunction and disjunction could be eliminated; it is merely an efficient way to talk about fusions. The unless-clauses are useful because they imply that the empty set is made true and false by the null state. This may seem strange but it has technical advantages. In particular, it allows us to say that fusing a state with the truth-makers or falsity-makers of the empty set simply returns the original state; for, $s \sqcup \square = s$.

It will also occasionally be useful to allow ourselves to ignore certain cardinality issues. To see what we mean, note that, given a countable infinity of states, there will be an uncountable infinity of propositions. Hence, we cannot express all propositions in a language. It is sometimes useful to ignore this complication, and we can do so by making the following assumption.[4]

Assumption 5. *Expressibility of propositions:* For any proposition $\langle V, F\rangle$, we can add sentences, $\Gamma \cup \Delta$, to our language such that every member of V contains exactly one truth-maker for each sentence in Γ and every member of F contains exactly one falsity-maker for each sentence in Δ.

With this assumption in place, if we encounter a proposition, we can assume that our language has (or can be given) the resources to express the proposition and to do so in a sufficiently fine-grained way, namely as fine-grained as the states in the proposition. As will become clear in the next two chapters, the assumption is best understood as the mirror image of the fact that our grasp of states is limited by our language.

We have now spelled out most of the details of our models. However, one crucial element is missing. So far, we have put no constraints on the possible states; they may be any subset of our states. Fine often imposes the following constraints, and they will become important below:

Downward-Closure: If $s \in S^{\Diamond}$ and $t \sqsubseteq s$, then $t \in S^{\Diamond}$.

Exclusivity: If $s \in |p|^{+}$ and $t \in |p|^{-}$, then $\forall u(s \sqcup t \sqcup u \notin S^{\Diamond})$.[5]

Exhaustivity: $\forall u \in S^{\Diamond}$, either $\exists s \in |p|^{+}$ $(u \sqcup s \in S^{\Diamond})$ or $\exists t \in |p|^{-}$ $(u \sqcup t \in S^{\Diamond})$.

Downward-Closure says that all parts of a possible state are possible. Exclusivity says that if you take any atomic[6] sentence, then if you fuse one

[4]A lot of the work of this assumption can be done by using canonical models (see Fine, 2017a, 647). This works, e.g., for completeness proofs. However, certain points can be seen more clearly with the assumption in place.

[5]This formulation differs from Fine's in the quantification over further states u. In the presence of Downward-Closure, the two formulations are equivalent.

[6]Stipulating these constraints for atomic sentences suffices (given the semantic clauses) to enforce them for the whole language.

of its verifiers with one of its falsifiers together with any other state, you always get an impossible state. And Exhaustivity says that if you have a possible state and an atomic sentence, then you can extend it to a possible state either by fusing it with a verifier of the sentence or by fusing it with a falsifier of the sentence.

It will emerge in due course that the three constraints of Downward-Closure, Exclusivity, and Exhaustivity correspond, respectively, to the structural rules of weakening, containment, and cut in a classical sequent calculus. Moreover, the semantic clauses correspond to our operational sequent rules. This correspondence will be at the center of the isomorphism between the theory from the previous chapters and truth-maker theory.

Before we get to this correspondence, however, we must address the perhaps central question in formulating any logical system: How should we define consequence? It is common in the literature to define consequence in truth-maker theory in non-modal terms, i.e., without any appeal to the distinction between possible and impossible states. In particular, Fine often uses the following notions of consequence:[7]

- *Entailment*: P entails Q iff every verifier of P is a verifier of Q (Fine, 2017a, 640-41).

- *Containment*: Q contains P iff (i) every verifier of Q includes as a part a verifier of P and (ii) every verifier of P is included as a part in a verifier of Q (Fine, 2017a, 640-41).

These notions can be extended to the case of multiple premises in different ways (Fine and Jago, 2018). As we will see below, however, an interesting equivalence emerges between truth-maker theory and the theory in the previous chapters if we adopt, instead, a different notion of truth-maker consequence, which we call "truth-maker consequence" or "TM-validity" and write as $\models_{TM}$.

[7]Fine takes the relata of consequence to be propositions. We will work with sentences. Assumption 5 ensures that this is unproblematic for our purposes.

Truth-maker consequence is a modal notion; it crucially involves the distinction between possible and impossible states. The idea behind truth-maker consequence is similar to Bueno and Shalkowski's modalism about logic, which says that B follows from A if and only if the conjunction of A and the negation of B is impossible (Bueno and Shalkowski, 2013, 11-12). In contrast to modalism, however, truth-maker consequence is formulated at a meta-theoretic level, namely in terms of truth-makers and falsity-makers. This allows us to avoid reference to logical connectives, like negation and conjunction. We could hence reformulate modalism thus: B follows from A if and only if any fusion of a truth-maker of A and a falsity-maker of B is an impossible state. If we generalize the notion to sets of sentences, we get:

> *Truth-Maker Consequence*: $\Gamma \models_{TM} \Delta$ (in a model) iff (in that model) any fusion of verifiers for each $\gamma \in \Gamma$ and falsifiers for each $\delta \in \Delta$ is an impossible state, i.e., $s \notin S^{\Diamond}$ for all $s = u \sqcup t$ such that $u \Vdash \Gamma$ and $t \dashv\!\!\mid \Delta$.[8]

This gives us a notion of consequence relative to models. And if we have a model that models reality in the right way, it will be consequence relative to that model that captures what follows from what in the right way. In specifying a particular model, we can choose which material inferences are TM-valid in that model. If we want to ensure, e.g., that "o is colored" follows from "o is green", all we have to do is to ensure that every fusion of a falsity-maker for "o is colored" and a truth-maker for "o is green" is an impossible state. Thus, truth-maker consequence is not logical consequence; it is not closed under uniform substitution of non-logical expressions. As already intimated, we are especially interested in these wider notions of consequence. We can, however, easily restrict truth-maker consequence

[8] At this point it is again convenient that Definition 4 ensures that we always have a state to evaluate, even if $\Gamma \cup \Delta = \emptyset$ or if Γ has no verifiers or Δ no falsifiers. In such cases the relevant state is $\blacksquare$.

to logical consequence (LTM-validity) by quantifying over models in the usual way.

> *Logical Truth-Maker Consequence*: $\Gamma \models_{LTM} \Delta$ iff, in all models, $s \notin S^{\Diamond}$ for all $s = u \sqcup t$ such that $u \Vdash \Gamma$ and $t \dashv\Vert \Delta$.

As we will see shortly, making different choices in the semantic set-up leads to different consequence relations for TM-validity and LTM-validity. Hence, these are really families of consequence relations. We will disambiguate where necessary by using appropriate labels. It will turn out that different members of these families correspond exactly to the different accounts of consequence from the previous chapter.

4.2 Articulating the Correspondence

Let us now bring out how the version of truth-maker theory from the previous section maps onto our sequent calculi from the previous chapter. To do so, it will be useful to start with a somewhat tweaked sequent calculus for classical logic and see how it maps onto the truth-maker theory above in the fully structural case. We will then explain the correspondence at a conceptual level. This will prepare us for relaxing structural rules in the next section.

Let the language $\mathfrak{L}$ be as above, and let's consider the following sequent calculus for classical propositional logic, which we call "CL".

Structural Rules of CL:

$$\frac{}{\Gamma, p \succ p, \Delta}\,[\text{ID}] \qquad\qquad \frac{\Gamma \succ \Delta}{\Theta, \Gamma \succ \Delta, \Sigma}\,[\text{weakening}]$$

$$\frac{\Gamma \succ \Delta, p \qquad p, \Gamma \succ \Delta}{\Gamma \succ \Delta}\,[\text{cut}]$$

Operational Rules of CL:

$$\frac{\Gamma, A, B \succ \Delta}{\Gamma, A \wedge B \succ \Delta}\,[\mathrm{L}\wedge] \qquad \frac{\Gamma \succ \Delta, A \qquad \Gamma \succ \Delta, B}{\Gamma \succ \Delta, A \wedge B}\,[\mathrm{R}\wedge]$$

$$\frac{\Gamma, A \succ \Delta \qquad \Gamma, B \succ \Delta}{\Gamma, A \vee B \succ \Delta}\,[\mathrm{L}\vee] \qquad \frac{\Gamma \succ \Delta, A, B}{\Gamma \succ \Delta, A \vee B}\,[\mathrm{R}\vee]$$

$$\frac{\Gamma \succ \Delta, A}{\Gamma, \neg A \succ \Delta}\,[\mathrm{L}\neg] \qquad \frac{\Gamma, A \succ \Delta}{\Gamma \succ \Delta, \neg A}\,[\mathrm{R}\neg]$$

We write $\Gamma \vdash_{CL} \Delta$ if $\Gamma \succ \Delta$ is provable in this calculus. The sequent calculus is sound and complete with respect to classical propositional logic, which we denote by $\models_{CL}$ (appendix: Proposition 24).

Some comments are in order. First, [weakening] is redundant; [ID] suffices for monotonicity. Second, [cut] can be eliminated, i.e., omitting [cut] doesn't change which sequents are derivable. Third, double-lines indicate that the rule licenses a move in both directions, i.e., the usual top-to-bottom direction but also the derivation of any of the top sequents from the bottom sequent. The bottom-to-top directions of the double-line rules are redundant; i.e., the operational rules in a single-line variant of CL are invertible (appendix: Proposition 22). The double-line rules will nevertheless be illuminating in our context.

CL is interesting for our purposes because the rules of CL correspond in a straightforward way to aspects of our truth-maker theory. To bring this out, let's start with the operational rules, which correspond to the semantic clauses in our truth-maker theory.

4.2.1 The Logical Connectives

Consider an application of [L∧], i.e., a move from $\Gamma, A, B \succ \Delta$ to $\Gamma, A \wedge B \succ \Delta$ or vice versa. This tells us that $\Gamma, A, B \vdash_{CL} \Delta$ iff $\Gamma, A \wedge B \vdash_{CL} \Delta$. That rule application is sound for LTM-validity just in case, in all models, when we take any state, s, that is a fusion of truth-makers for each member of Γ and falsity-makers for each member of Δ, then all fusions of s with a truth-maker of A and a truth-maker of B are impossible iff all fusions of s with any truth-maker of $A \wedge B$ are impossible. Now, the semantic clause for the truth-makers of conjunctions says that the truth-makers of $A \wedge B$ are all and only the fusions of a truth-maker of A and a truth-maker of B. That is, the state r is a truth-maker of $A \wedge B$ iff $\exists u, t\,(u \Vdash A$ and $t \Vdash B$ and $r = u \sqcup t)$. Thus, the states that our sequent rule requires to be co-impossible when fused with s in all models are precisely those states that our semantic clause requires to be identical in all models. Because these states are identical, the results of fusing them with s are also identical. So, these fusions cannot differ in whether they are possible or impossible. Hence, $\Gamma, A, B \models_{LTM} \Delta$ holds iff $\Gamma, A \wedge B \models_{LTM} \Delta$ holds (and this obviously also holds for nonlogical truth-maker validity). So, our semantic clause (and+) is sufficient to ensure that [L$\wedge$] is valid for truth-maker consequence.

Moreover, our semantic clause (and+) is also necessary for the validity of [L$\wedge$]. For, suppose that there was a state, t, that was a truth-maker of $A \wedge B$ but not a fusion of a truth-maker of A and a truth-maker of B, then we could find a model in which $s \sqcup t$ is possible while fusing s with any truth-maker of A and any truth-maker of B is impossible. So, in that model, $\Gamma, A, B \models_{TM} \Delta$ but $\Gamma, A \wedge B \not\models_{TM} \Delta$. And we can generalize this strategy to cover all models, so that we have a counterexample to [L$\wedge$] for LTM-validity.

What this means is that our semantic clause for the truth-makers of conjunctions, namely (and+), and our left-rule for conjunction in the sequent calculus, namely [L$\wedge$], have the same impact on the consequence relation. They are equivalent in their significance for the consequence relations defined in the two frameworks.

The analogous claims hold of the other operational rules of CL. We can formulate this succinctly if we say that, given a model, a sequent deems exactly those states impossible that must be impossible for the sequent to hold.

Definition 6. *Deeming impossible*: A sequent $\Gamma \succ \Delta$ deems all and only those states impossible that are a fusion of verifiers for everything in Γ and falsifiers for everything in Δ, i.e., any state $s = u \sqcup t$ such that $u \Vdash \Gamma$ and $t \dashv\!\vert\vert \Delta$.

All of the operational rules of CL are such that the states that are deemed impossible by the conclusion sequent are already deemed impossible by a premise sequent. In fact, we can be more precise.

Lemma 7. *For every top-to-bottom application of an operational rule of* CL*, the set of states deemed impossible by the bottom-sequent is the union of the sets of states deemed impossible by the top-sequents (appendix: Lemma 26).*

The reason why this holds is that the states that the active sentence in the bottom-sequent contributes to the fusions that our bottom-sequent deems impossible are exactly the states that the active sentences in the top-sequents contribute to their fusions. To see what we mean, consider [R∧] as another example. Let s again be an arbitrary state that is a fusion of truth-makers for each member of Γ and falsity-makers for each member of Δ. The bottom-sequent, $\Gamma \succ A \wedge B, \Delta$, holds iff any fusion of such an s with any falsity-maker of $A \wedge B$ is impossible. Now, our semantic clause for the falsity-makers of conjunctions, namely (and$-$), tells us that the falsity makers of $A \wedge B$ are exactly those states that are either a falsity-maker of A or a falsity-maker of B. But those are precisely the kinds of state whose fusion with s is deemed impossible, respectively, by the two top-sequents of [R∧].

It follows that, in the bottom-to-top applications of the rules, we move from a sequent that deems certain states impossible to a sequent that deems only a part of these states impossible. So these rule-applications are also guaranteed to be valid. Hence, our operational rules cannot lead from TM-valid sequents to TM-invalid sequents. So our sequent rules are sound for TM-validity.

Proposition 8. *All operational rules of* CL *are sound for* $\models_{TM}$*, i.e., if all the top-sequents of a rule-application are TM-valid, then so is the bottom sequent (appendix: Proposition 27).*

Note that like the lemma on which it rests, this result is independent of any constraints on possible states. The semantic clauses for the connectives suffice to ensure the result. Indeed, there is a perfect match between the semantic clauses and the sequent rules. The clauses for truth-makers correspond to the left-rules because sentences on the left of a sequent contribute their truth-makers to the states that the sequent deems impossible. And the clauses for falsity-makers correspond to the right-rules because sentences on the right side contribute their falsity-makers to the state that the sequent deems impossible.

4.2.2 The Structural Rules

This brings us to the structural rules. We won't consider permutation and contraction because we are working with sets. So, we have just three structural rules: [ID], [weakening], and [cut]. These are the sequent rules that correspond to (versions of) the reflexivity, monotonicity, and transitivity of consequence. They are the constraints that allow our consequence relations to encode only closed reason relations. If we want to be able to theorize open reason relations, we must relax these constraints. Hence, if we want to formulate versions of truth-maker theory that can capture open reason relations, it is crucial for us to understand what corresponds to these structural constraints in truth-maker theory and how we can relax them.

As already intimated, our three structural rules correspond exactly to our three constraints on possible states. Exclusivity corresponds to [ID]. Downward-Closure corresponds to [weakening]. And Exhaustivity corresponds to [cut]. Let us quickly run though the arguments for the three rules.

Recall that Exclusivity says that if $s \in |p|^+$ and $t \in |p|^-$, then $\forall u(s \sqcup t \sqcup u \notin S^\Diamond)$, i.e., if you fuse a truth-maker and a falsity-maker of an atomic sentence with any other state, then the result is always an impossible state. If that holds in a model, then clearly $\Gamma, p \mathrel{|\!\overline{\underline{}}}_{TM} p, \Delta$ in that model. And if it holds in all models, then $\Gamma, p \mathrel{|\!\overline{\underline{}}}_{LTM} p, \Delta$. But that is precisely what the [ID] rule allows us to prove; for it allows us to derive $\Gamma, p \succ p, \Delta$.

Downward-Closure says that if $s \in S^\Diamond$ and $t \sqsubseteq s$, then $t \in S^\Diamond$, i.e., if a state is possible, then all of its parts are possible. So, if a state is impossible, then all the states of which it is a part are impossible. A sequent $\Gamma \succ \Delta$ is TM-valid, in a model, just in case all the states that are fusions of truth-makers of Γ and falsity-makers of Δ are impossible. If that is the case and Downward-Closure holds, then all the states that include such a fusion as a part are also impossible. Now, such a fusion is part of every state that is deemed impossible by a sequent of the form $\Theta, \Gamma \succ \Delta, \Sigma$. For these states are fusions of truth-makers of Γ and falsity-makers of Δ with some further states, namely truth-makers of Θ and falsity-makers of Σ. Hence, if $\Gamma \succ \Delta$ is TM-valid, then so is $\Theta, \Gamma \succ \Delta, \Sigma$. Thus, Downward-Closure is sufficient for the soundness of [weakening].

Downward-Closure is also necessary for the soundness of [weakening] if we assume the expressibility of propositions. For suppose that Downward-Closure fails, so that there is an impossible state, s, that is part of a possible state, $s \sqcup t$. If we can find a sequent, $S \succ$, that deems only the impossible state, s, impossible and also a sentence, T, whose only truth-maker is t, then $S \mathrel{|\!\overline{\underline{}}}_{TM} \varnothing$ but $S, T \not\mathrel{|\!\overline{\underline{}}}_{TM} \varnothing$. And that gives us a counterexample to [weakening]. Hence, Downward-Closure and [weakening] are equivalent regarding their effects on the consequence relations in their respective frameworks.

Lastly, remember that Exhaustivity says that if a state, u, is possible, then either $\exists s \in |p|^+ \ (u \sqcup s \in S^\Diamond)$ or $\exists t \in |p|^- \ (u \sqcup t \in S^\Diamond)$. That is, for any atomic sentence, a possible state can always be extended to another possible state by fusing it either with a truth-maker or with a falsity-maker of the atomic sentence. To see that this corresponds to [cut], it is helpful to read [cut] contrapositively: if $\Gamma \not\succ \Delta$, then either $\Gamma \not\succ p, \Delta$ or $\Gamma, p \not\succ \Delta$. If we think of this in terms of TM-validity, it says that if there is a possible fusion of truth-makers of Γ and falsity-makers of Δ, then there is either a possible fusion of truth-makers of $\Gamma \cup \{p\}$ and falsity-makers of Δ or there is a possible fusion of truth-makers of Γ and falsity-makers of $\Delta \cup \{p\}$. Exhaustivity ensures that this holds; so it ensures that [cut] is sound.

Moreover, assuming the expressibility of propositions again, Exhaustivity is not only sufficient but also necessary for the soundness of [cut]. For suppose that there is a possible state, u, such that neither $\exists s \in |p|^+ \ (u \sqcup s \in S^\Diamond)$ nor $\exists t \in |p|^- \ (u \sqcup t \in S^\Diamond)$. If u is the only truth-maker of U and we let u be possible, then $U \not\models_{TM} \emptyset$ but $U, p \models_{TM} \emptyset$ and $U \models_{TM} p$. And that is a counterexample to [cut]. Hence, Exhaustivity and [cut] are equivalent with respect to their effect on consequence.

So, not only do the operational rules of our sequent calculus correspond exactly to the semantic clauses in truth-maker theory but the structural rules correspond exactly to constraints on possible states. We can summarize our results regarding the correspondence between structural rules and constraints on possible states in the following proposition.

Proposition 9. *The rules [ID], [weakening], and [cut] preserve TM-validity iff possible states obey Exclusivity, Downward-Closure, and Exhaustivity respectively (appendix: Proposition 25).*

Putting these results together suffices to show that $\models_{LTM}$ coincides with classical propositional consequence, because CL is sound and complete with respect to $\models_{LTM}$.

Proposition 10. *If we impose Exclusivity, Exhaustivity, and Downward-Closure, then* $\Gamma \models_{LTM} \Delta$ *iff* $\Gamma \vdash_{CL} \Delta$ *iff* $\Gamma \models_{CL} \Delta$ *(appendix: Proposition 29).*

Of course, our goal here is not to recover classical logic in truth-maker theory. Rather, the interest of this result lies in the fact that we have obtained it by finding a correspondence between the parts of our truth-maker theory and our sequent calculus. In the next subsection, we will bring out this correspondence at a more conceptual and less technical level. Once we have a clearer picture of how truth-maker theory and the sequent calculus rules from the previous chapter correspond to each other, this will enable us to formulate versions of truth-maker theory that allow us to codify open reason relations. For, we can then drop the principles that correspond to [cut] and [weakening]. As already intimated, those will be Exhaustivity and Downward-Closure.

4.2.3 Two Sides of One Coin

Given the correspondence between operational rules and the semantic clauses in truth-maker theory and the structural rules and constraints on possible states, we can now see how the pragmatics-first perspective that we presented in the previous two chapters has a mirror image in a version of the semantics-first alternative perspective, namely the version of truth-maker theory presented above.

According to the pragmatics-first perspective that we have presented in the first chapter, what it means to say that Δ follows from Γ is that if one is committed to assert everything in Γ, then one cannot be entitled to deny everything in Δ. The commitments to the elements of Γ are, in this sense, normatively incompatible with denying everything in Δ. This is a version of Restall's normative bilateralism. It differs from Restall's version only in the articulation of the normative incompatibility as a preclusion from entitlement, whereas Restall uses a primitive notion of "out-of-boundness" that applies to positions, i.e., combinations of assertions and denials.

Normative bilateralism offers an intuitive interpretation of sequents and sequent rules, and Restall's notions of positions being in-bounds or out-of-bounds is useful here. That the sequent $\Gamma \succ \Delta$ is valid, e.g., can be understood as the claim that any position that is a combination of assertions of everything in Γ and denials of everything in Δ is out-of-bounds. And sequent rules can then be interpreted as telling us that certain kinds of positions are out-of-bounds if other kinds of positions are out-of-bounds. We can now give exactly parallel interpretations of sequents and sequent rules in terms of truth-maker theory: according to truth-maker consequence, what it means to say that Δ follows from Γ is that any state that is a fusion of truth-makers for everything in Γ and falsity-makers for everything in Δ is impossible. And sequent rules tell us that if certain kinds of fusions are impossible, then other kinds of fusions are also impossible.

Notice the common structure, which comprises three aspects: First, we have elements that are picked out by sentences and that can play two kinds of role. On the pragmatic side, we have assertions and denials of sentences. On the semantic side, we have truth-makers and falsity-makers of sentences. Second, we have a notion of combining these elements into larger structures. On the pragmatic side, we have combinations of assertions and denials, which we call "positions." And on the semantic side, we have fusions of truth-makers and falsity-makers. Third, we distinguish two classes of such combinations. On the pragmatic side, we distinguish between in-bounds and out-of-bounds positions. And on the semantic side, we distinguish between possible and impossible fusions of states.

The two distinctions between two kinds of combinations are both modal distinctions, but one modality is normative and the other alethic. On the pragmatic side, our elements are acts of concept use, and we divide their combinations into the normatively proper ones and the improper ones (by the lights of the relevant conceptual norms). On the semantic side, our elements are the states that our concept use represents, i.e. representanda, and we divide their combinations (fusions) into the alethically possible ones and alethically impossible ones (according to the relevant kind of alethic modality). Thus, we have a normative modality governing acts of concept use on the one side, and an alethic modality governing worldly states on the other side.

Let us bring out this correspondence explicitly by looking at the structural and operational rules of our sequent calculus again. If we use an additive version of the cut-rule (i.e. CT), we can formulate the normative bilateralist (NB) and the truth-maker bilateralist (TM) interpretations of cut thus:

NB-CT: For any in-bounds position and any sentence, A, one can extend the position to an in-bounds position either by asserting or by denying A.

TM-CT: For any possible state and any sentence, A, one can extend the state into a possible state by fusing it with either a verifier or a falsifier of A.

For Containment, the two interpretations are the following:

NB-CO: Any position in which any sentence is asserted and also denied is out-of-bounds.

TM-CO: Any state that includes a verifier and also a falsifier for any sentence is impossible.

And here are the two interpretations of Monotonicity.

NB-MO: All positions that include an out-of-bounds position are themselves out-of-bounds.

TM-MO: All states that include an impossible state are themselves impossible.

We thus have two interpretations of the structural principles that yield closed reason relations. We have a normative-pragmatic interpretation of what it means for reason relations to be closed, and we now also have an alethic-modal interpretation. More precisely, if an alethic modality obeys the principles TM-CT, TM-CO, and TM-MO, then it defines a closed reason relation over sentences that represent the states that the modality governs.

We can also give two interpretations of our operational rules. For the left-rules, we can provide parallel interpretations in the following way:

NB-left: Left-rules specify the contributions that the assertions of complex sentences make to positions being out-of-bounds in terms of the contributions made by the assertions or denials of their constituent sentences.

TM-left: Left-rules specify the contributions that the verifiers of complex sentences make to states being impossible in terms of the contributions made by the verifiers or falsifiers of their constituent sentences.

And the parallel formulations for the right-rules are as follows:

NB-right: Right-rules specify the contributions that the denials of complex sentences make to positions being out-of-bounds in terms of the contributions made by the assertions or denials of their constituent sentences.

TM-right: Right-rules specify the contributions that the falsifiers of complex sentences make to states being impossible in terms of the contributions made by the verifiers or falsifiers of their constituent sentences.

We thus have two parallel interpretations of valid sequents and sequent rules: one alethic-modal, the other pragmatic-normative. In both interpretations, the consequence relation holds between two sets just in case certain states or positions are ruled out. The modality of this "ruling out" is normative in one case and alethic in the other case.

How do these two kinds of "ruling out" relate to each other? In the ideal case in which our norms of concept use are flawless, they correspond exactly to the impossibility of states. That is, the relation between the normative-pragmatic and the alethic-modal relations is itself normative. Our norms of concept use are defective unless the combinations of assertions and denials to which they say no one can be entitled match the states that are impossible. Hence, in the ideal case, the two structures are isomorphic.

As already intimated, we will later identify the roles that the elements of these structures play in them with rational forms, which are the forms that are shared between discursive activities and the worldly states that these discursive activities are about. Before we do so, however, we have to make sure that the correspondence between truth-maker theory and our normative pragmatics doesn't break down once we allow for open reason relations.

4.3 Accommodating Open Reason Relations

We have seen in the previous section that the structural rules of cut and monotonicity correspond to constraints on possible states that are familiar from Fine, namely Exhaustivity and Downward-Closure.[9] This suggests that if we want to allow for reason relations that are open in the sense that

[9] We also saw that Containment corresponds to Exclusivity. Since rejecting Containment isn't of much importance for us, however, we will only mention it in passing. In the previous chapter, we also talked about Contraction. Giving up Contraction in truth-maker theory would require that we define fusion not as a least upper bound. We won't

they are not constrained by cut or monotonicity, we can do that in truth-maker theory by dropping Exhaustivity or Downward Closure. The aim of this section is to show that this is indeed the case.

4.3.1 Nontransitive Truth-Maker Consequence

Let us then formulate a nontransitive version of truth-maker consequence by dropping Exhaustivity. Now, this change by itself won't have any effect on our logical consequence relation because cut is admissible. In terms of truth-maker theory: Exhaustivity is redundant if Exclusivity is the only constraint on possible states that holds in all models. There is, however, a well-known way to make cut fail in a useful and interesting way, namely the so-called strict/tolerant logic STT, which includes a transparent truth-predicate and was developed as a response to the semantic paradoxes (Cobreros et al., 2013, 2012; Ripley, 2012). Hence, recovering STT in truth-maker theory doesn't only give us a new kind of semantics for nontransitive logics but is also an interesting test-case for our idea that we can allow for nontransitive reason relations in truth-maker theory by dropping Exhaustivity.[10]

The logic STT is especially interesting for us because Ripley (2013; 2015) endorses this logic as a response to the paradoxes on the basis of normative bilateralism. If we consider rejecting the principle NB-CT, this tells us what it means, given normative bilateralism, to reject cut. To reject NB-CT is to hold that there are in-bounds positions and sentences, such that adding an assertion of the sentence to the position will make it out-

pursue this here, but we will generalize the insights from the current chapter in the next one, where the issue of Contraction will come up again.

[10]This kind of semantics for STT was first developed and presented in (Hlobil, 2022a) and (Hlobil, 2022b). In these papers, there are hints at the greater power of truth-maker semantics relative to the usual strong Kleene semantics for STT. The treatment of non-monotonic consequence relations in this framework that we are offering below makes good on the promissory notes in these papers.

of-bounds but adding a denial of the sentence will also make the position out-of-bounds. That is, it can happen that one can neither coherently assert nor coherently deny a sentence. Or to put it in terms of commitment and entitlement, commitment to the sentence precludes one from being entitled to reject anything, but one is also precluded from rejecting the sentence. Thus one can neither reasonably commit to the sentence nor can one reject it. According to the nontransitive approach to paradox, this is the case for paradoxical sentences like the Liar sentence.

To spell out some of the formal details of this non-transitive approach, let's add to our object language a canonical name $\bar{A}$ for every sentence A and a truth-predicate, *Tr*, for which we add the following sequent rules to CL.[11]

$$\frac{\Gamma, A \succ \Delta}{\Gamma, Tr(\bar{A}) \succ \Delta}\,[\mathrm{Lt}] \qquad\qquad \frac{\Gamma \succ A, \Delta}{\Gamma \succ Tr(\bar{A}), \Delta}\,[\mathrm{Rt}]$$

Under the interpretation of normative bilateralism, these rules stipulate that asserting A and asserting $Tr(\bar{A})$ always have the same effect on whether a position is out-of-bounds, and the same holds for their denials.

Let us now allow for self-reference by allowing sentences that include their own names. Thus, we can formulate a Liar sentence, $\neg Tr(\bar{\lambda})$, whose name is $\bar{\lambda}$. This sentence says of itself that it is not true. Note that λ is everywhere intersubstitutable with $\neg Tr(\bar{\lambda})$ *salva consequentia*.[12] Since $Tr(\bar{\lambda})$ is an atomic sentence, [ID] yields $Tr(\bar{\lambda}) \succ Tr(\bar{\lambda})$. Using [Lt], [Rt], our negation rules, and the intersubstitutability of $\neg Tr(\bar{\lambda})$ and λ, we can

[11] As before, we leave out quantifiers, thus restricting us, in effect, to pure predicate logic (i.e., predicate logic without quantifiers or identity). The truth-predicate and the name of the liar sentence are the only things we really need from the language of the predicate calculus.

[12] Here, we stipulate means of self-reference by fiat: we let λ and $\neg Tr(\bar{\lambda})$ simply be identical. This allows us to avoid the complications of adding self-reference via Gödel numbers. See Ripley (2012, 355) for more details.

derive $\succ \lambda$ and $\lambda \succ$ (Ripley, 2013). Applying [cut] now yields the empty sequent. Something has to give.

Ripley argues that normative bilateralism offers a motivation for rejecting cut. For it is intuitively plausible that adding an assertion of the liar sentence or a denial of the Liar sentence to any in-bounds position makes the position out-of-bounds (Ripley, 2013, 152). Asserting and denying the Liar sentence are both normatively ruled out. In working out this idea, Ripley and others have formulated the non-transitive logic STT that includes a transparent truth-predicate and whose consequence relation includes every classically valid inference. Given a language with a truth-predicate, (the propositional fragment of) STT can be formulated proof-theoretically by adding [Lt] and [Rt] to CL while deleting [cut].

Definition 11. STT-calculus: The STT-calculus is the sequent calculus that is like CL except that it includes the rules [Lt] and [Rt] and doesn't include the rule [cut]. We say that $\Gamma \models_{\mathsf{STT}} \Delta$ iff the sequent $\Gamma \succ \Delta$ is derivable in this calculus.

We now make the analogous changes in our truth-maker theory. The idea behind a transparent truth-predicate is that A and $Tr(\bar{A})$ play the same role (at least in extensional contexts) with respect to consequence. In truth-maker theory, we can get the corresponding effect by stipulating that these two sentences have the same truth-makers and the same falsity-makers.

(tr+) $\quad s \Vdash Tr(\bar{A})$ iff $s \Vdash A$

(tr−) $\quad s \dashv\!| Tr(\bar{A})$ iff $s \dashv\!| A$

Given these semantic clauses for our truth-predicate, what can we say about the Liar sentence? Since we defined λ as $\neg Tr(\bar{\lambda})$, they must have the same truth-makers and falsity-makers. This would trivialize $\models_{TM}$ in the presence of Exhaustivity (appendix: Proposition 31). But we can allow for failures of Cut by dropping Exhaustivity.

Definition 12. Let $\models^{STT}_{LTM}$ be the consequence relation just like $\models_{LTM}$ except that we include models that violate Exhaustivity and the language has a truth-predicate whose interpretation obeys the clauses (tr+) and (tr−).

These changes have exactly the effects that we have envisaged. The semantic clauses have the same effect on consequence as the sequent rules for the truth-predicate, and dropping [cut] and Exhaustivity have also the same effect. Hence, this version of (logical) truth-maker consequence coincides with the consequence relation of STT.

Proposition 13. $\Gamma \vdash_{\mathsf{STT}} \Delta$ *iff* $\Gamma \models^{STT}_{LTM} \Delta$ *(appendix: Proposition 36).*

Notice that the two formal systems don't only match in their consequence relations. Rather, they match in a piece-by-piece fashion, in the way explained in the previous subsection. Indeed, all we did was to use the correspondence from the previous subsection to formulate a version of truth-maker consequence that mirrors the sequent calculus for STT.

We can illustrate this by spelling out how the philosophical interpretation of failures of cut can be translated from normative bilateralism into truth-maker theory. For normative bilateralism, the intuitive idea behind the rejections of cut was that any position that include an assertion or a denial of the Liar sentence is out-of-bounds. When we translate this into truth-maker bilateralism, the result is this: Any state that includes a verifier or a falsifier of the liar sentence is impossible. Given any possible state, this yields a violation of Exhaustivity. Since the world is a possible state, we can express the idea by saying that the world cannot contain anything that makes the liar sentence either true or false. Just as we should neither assert nor deny the liar sentence, so the world can neither verify nor falsify it.

We have thus formulated a truth-maker semantics for strict/tolerant logic, and this illustrates how we can allow for nontransitive reason relations in truth-maker theory.[13] Now, the reason why we rejected transitivity in the previous chapters had nothing to do with the semantic paradoxes. We argued rather that once we free ourselves from the constraint of monotonicity on reason relations, our language will have crippling expressive limitations unless we also allow for nontransitive reason relations. Hence, we must consider whether the strategy for allowing for open reason relations in truth-maker theory can yield the desired results when we move away from STT as a test-case.

4.3.2 Nonmonotonic Truth-Maker Consequence

In the previous subsection, we have explored the results of rejecting transitivity as requirements on our consequence relation. And we illustrated how we can allow for open reason relations in truth-maker theory by recovering STT in the framework of truth-maker consequence. We now want to make the final step to show that the framework of truth-maker consequence is isomorphic to the logic that we presented in the previous chapter.

We saw above that monotonicity corresponds to Downward-Closure in truth-maker consequence. Hence, in order to go non-monotonic in truth-maker consequence, we must allow failures of Downward-Closure. We have argued in the first chapter that if we free ourselves from the constraint of monotonicity, then we should also reject transitivity. And the logics we presented in the second chapter allowed for failures of both structural principles. In order to mirror the developments from the pre-

[13]There are well-known relations between ST and the logics LP, K3, and TS (Dicher and Paoli, 2019; Barrio et al., 2015). These relations can be spelled-out in truth-maker theory. If we do that, LP emerges roughly as the logic of impossible falsity-makers. K3 emerges roughly as the logic of possible truth-makers. And TS emerges as the logic that rejects Exclusivity. We will return to these connections in a more general setting in the next chapter.

vious chapter within truth-maker theory, we will hence not only drop Downward-Closure but also Exhaustivity.

So far, this chapter focused on logical consequence. Recall, however, that our motivation for thinking about open reason relations was that material reason relations violate monotonicity. That is, *material* implications and imcompatibilities can be defeated by adding more premises (or conclusions). Thus, our goal is not to formulate a nonmonotonic logic but rather to formulate nonmonotonic consequence relations, to which we can then add logical vocabulary and extend the consequence relation accordingly. Indeed, that we should start with material, non-logical consequence relations is a core commitment of logical expressivism. For, the aim of logical expressivism is to understand logical vocabulary in terms of its expressive job with respect to antecedent, material, non-logical relations of implication and incompatibility.

Putting the points of the previous two paragraphs together, we want to formulate nonmonotonic and nontransitive consequence relations that include material consequences within truth-maker theory. Moreover, we want to do so in a way that brings out the correspondence between our systems from the previous chapter and truth-maker consequence.

Recall that we started in the previous chapter with a material base consequence relation, i.e., a consequence relation over atomic sentences. And we thought of this consequence relation as encoding simultaneously the material implications and incompatibilities between our atomic sentences. In other words, such base consequence relations encode non-logical reason relations of both fundamental varieties: reasons-for and reasons-against. With $\mathfrak{L}_0$ being our non-logical language, such a material base consequence relation is simply a relation between sets of atomic sentences.

Definition 14. *Material Base*: A material base, $\mathfrak{B}$, is a relation between sets of atomic sentences, i.e., $\mathfrak{B} \subseteq \mathcal{P}(\mathfrak{L}_0) \times \mathcal{P}(\mathfrak{L}_0)$. A material base obeys Containment iff $\langle \Gamma_0, \Delta_0 \rangle \in \mathfrak{B}$ whenever $\Gamma_0 \cap \Delta_0 \neq \emptyset$.

We say that the material base includes a sequent $\Gamma_0 \succ \Delta_0$ iff $\langle \Gamma_0, \Delta_0 \rangle \in \mathfrak{B}$. To illustrate, let's suppose that $c =$ "This is a chair" implies $s =$ "You can sit on this." Hence, the pair $\langle \{c\}, \{s\} \rangle \in \mathfrak{B}$. Since this implication is defeated if we add the additional premise $b =$ "This is broken," the corresponding pair is not in the material base. That is, $\langle \{c, b\}, \{s\} \rangle \notin \mathfrak{B}$. Similarly, since "This is a chair" is incompatible with $v =$ "This is a violin," we have $\langle \{c, v\}, \varnothing \rangle \in \mathfrak{B}$. But since this incoherence is cured by adding $a =$ "This is part of an art project that makes pieces of furniture that are musical instruments," we say $\langle \{c, v, a\}, \varnothing \rangle \notin \mathfrak{B}$. Thus, our two reason relations, implication and incompatibility, are both defeasible, and we can model this in our formalism with an appropriate material base consequence relation. This is all familiar from the previous chapters.

Moreover, we showed in the previous chapter how we can add logical vocabulary to such a material base, namely by closing the material base under the operational rules of CL.

Definition 15. $\mathsf{NM}_{\mathfrak{B}}$ is the sequent calculus that is like CL except that it has no structural rules and the axioms of $\mathsf{NM}_{\mathfrak{B}}$ are the sequents in the material base $\mathfrak{B}$. We say that $\Gamma \mathrel{|\!\!\underset{NM}{\overset{\mathfrak{B}}{\sim}}} \Delta$ iff the sequent $\Gamma \succ \Delta$ is derivable in $\mathsf{NM}_{\mathfrak{B}}$.

We saw in the previous chapter that if we restrict our material bases to those that obey Containment, all consequence relations defined by such a calculus include all of classical logic. And we showed how to add various kind of object language operators to make explicit, in the object language, various local features of such consequence relations.

Our question now is how we can find truth-maker formulations of these consequence relations. The first step is to ensure that a truth-maker consequence relation includes all the atomic sequents in a given material base $\mathfrak{B}$. We can do that by stipulating, for every pair $\langle \Gamma_0, \Delta_0 \rangle \in \mathfrak{B}$, that any fusion of verifiers for everything in Γ_0 and falsifiers for everything in Δ_0 is impossible. That is what the following definition does.

Definition 16. $\mathrel{|}\!\!\!\overset{NM_{\mathfrak{B}}}{\underset{TM}{=\!=}}$ is the consequence relation that is just like $\models_{TM}$ except that the only constraint on possible states is that $s \notin S^{\Diamond}$ iff $s = \bigsqcup\{g_1, ..., g_n, d_1, ..., d_m\}$ such that $\langle \Gamma_0, \Delta_0 \rangle \in \mathfrak{B}$ and $\Gamma_0 = \{\gamma_1, ..., \gamma_n\}$ and $\Delta_0 = \{\delta_1, ..., \delta_m\}$ and $\forall 1 \leq i \leq n\,(g_i \in |\gamma_i|^+)$ and $\forall 1 \leq i \leq m\,(d_i \in |\delta_i|^-)$.

This definition says that, for every sequent in the material base, every state that the sequent deems impossible is impossible in our models. Note that we don't really need to quantify over models in order to define $\mathrel{|}\!\!\!\overset{NM_{\mathfrak{B}}}{\underset{TM}{=\!=}}$. The model in which all and only those states are impossible that must be impossible for the material base to hold already includes all the counterexamples to all invalid sequents.

If we understand consequence in this way, then the failures of monotonicity in the examples above look as follows: In the absence of any defeating facts, the state of something being a chair and the state of one being unable to sit on it are incompatible, i.e., their fusion is impossible. However, if we add to this fusion the state that the object in question is broken, then the states are no longer incompatible, i.e., the fusion of all three states is a possible state. Similarly, the states of something being a chair and the state of it being a violin are incompatible, i.e., their fusion is an impossible state. However, if we use this state with the state of the object being part of an art project that makes pieces of furniture that are musical instruments, then the resulting state isn't impossible. The thought is that certain states can make states compatible with each other that are otherwise incompatible. These states need, as it were, the further state in order to fit together.

How can we add logical vocabulary to such a truth-maker model? In the same way as above. We add disjunctions, conjunctions, and negations to our language in the usual way, and we assign them truth-makers and falsity-makers in accordance with the semantic clauses above. The truth-makers of conjunctions are the fusions of truth-makers for each conjunct, etc., the falsity-makers of negations are the truth-makers of the negata, etc.,

etc. The states and their modal states as possible and impossible remains unchanged.

We know that the semantic clauses in truth-maker theory correspond to the operational rules in sequent calculi. So adding logical vocabulary that obeys the semantic clauses has the same effect on consequence as closing the base sequents under the corresponding sequent rules. And since we didn't enforce Exhaustivity and Downward-Closure but insisted on Exclusivity, the resulting consequence relation is guaranteed to obey Containment but may be nontransitive and nonmonotonic. In fact, the result is exactly what you might expect: the sequent calculus version of our current logic and the truth-maker consequence relation of it coincide if the material bases are the same.

Proposition 17. $\Gamma \mathrel{\vert\!\sim^{\mathfrak{B}}_{NM}} \Delta$ *iff* $\Gamma \models^{NM_{\mathfrak{B}}}_{TM} \Delta$ *(appendix: Proposition 37).*

So we now have a way to construct a truth-maker semantics for any nonmonotonic logic that can be obtained by closing an arbitrary set of atomic sequents under given operational rules. Thus, we can formulate our logics from the previous chapter within truth-maker theory in a surprisingly straightforward way. We simply take the semantic clauses that correspond to the desired sequent rules, and we stipulate the material base by way of a constraint on possible states. If we restrict ourselves to bases that obey Containment, then the result will include all classically valid sequents. If we want to enforce a structural rule, we add the corresponding constraint on possible states. If we want to add another bit of logical vocabulary, we read off the semantic clauses from the sequent rules. The clause for truth-makers is given by the left rule, and the clause for falsity-makers is given by the right rule. We saw an example of this when we looked at the clauses for the truth-predicate.

4.4 Rational Forms

We have now seen how the sequent calculus definitions of consequence relations from the previous chapter can be recast in truth-maker theory. We can not only give semantic theories for which our sequent calculi are sound and complete but there is also a piece-by-piece correspondence between the two kinds of formalism. The structure that is shared between these two frameworks we shall call the structure of rational forms. In order to bring this out more clearly, we will start with an interlude regarding the Aristotelian roots of our ideas. We then relate them to the isomorphism that we spelled out above.

4.4.1 Interlude: Aristotelian Roots

The similarity between our term "rational form" and the traditional term "intelligible form" is, of course, no accident (though there are obviously important differences). The ideas in this chapter are recognizably Aristotelian in spirit. Let us make this historical connection explicit.

In the *Categories*, Aristotle notes that assertions and denials are incompatible with each other in a way that mirrors the way in which the facts that they assert or deny are incompatible.

> [W]hat underlies an affirmation or negation [is not] itself an affirmation or negation. For an affirmation is an affirmative statement and a negation a negative statement, whereas none of the things underlying an affirmation or negation is a statement. These are, however, said to be opposed to one another as affirmation and negation are; for in these cases, too, the manner of opposition is the same. For in the way an affirmation is opposed to a negation, for example "he is sitting"—"he is not sitting", so are opposed also the actual things underlying each, his sitting—his not sitting. (Aristotle, Cat. 12b5-b16)

Translated into our framework, what Aristotle is saying is that an assertion of "he is sitting" plays the same role within the normative-pragmatic structure that a state that makes the sentence true is playing in the alethic-modal structure. The states that make a sentence true are opposed to the states that make a sentence false in a way that corresponds to the way in which the assertion of the sentence is opposed to its denial. That doesn't mean that the truth-maker of an assertion—what underlies the assertion—is an assertion. It merely means that they play the same role.

We started with a pragmatics-first perspective, and we took as our paradigms of contentful items speech acts. We suggested that these items are contentful in virtue of the conceptual norms that govern their use, and we have now seen that these norms are, in the ideal case, isomorphic to the possibility of states. For Aristotle, however, the primary contentful items are thoughts. So, for him, the question is in virtue of what a particular thought is about a particular aspect of the world. What is it for a thought to be a thought that things are thus-and-so? Shields (2020, sec. 7) helpfully summarizes Aristotle's answer as follows:

> *S* thinks *O* if and only if: (i) *S* has the capacity requisite for receiving *O*'s intelligible form; (ii) *O* acts upon that capacity by enforming it; and, as a result, (iii) *S*'s relevant capacity becomes isomorphic with that form.

Let's put aside aspects (i) and (ii) for now, and let's focus on (iii), i.e., the isomorphism of a thought and its object. Aristotle holds that for the intellect to think something is for it to take on the intelligible form of the (potential) part of the world of which it is thinking. However, while in the thing thought about the form informs the matter of which the world around us consists, when we think about the thing, this form informs our intellect. So, the form is identical but what is informed differs.[14] The intel-

[14]They don't differ in the special case in which what is thought about doesn't have any matter, i.e., in the case of abstract objects. "For in the case of objects which involve no

lect and what it thinks about are hence isomorphic in the sense of sharing there form. So when you think, e.g., about a stone, your intellect takes on the form of the stone.

> Within the soul the faculties of knowledge and sensation are potentially these objects, the one what is knowable, the other what is sensible. They must be either the things themselves or their forms. The former alternative is of course impossible: it is not the stone which is present in the soul but its form. It follows that the soul is analogous to the hand; for as the hand is a tool of tools, so thought is the form of forms and sense the form of sensible things. (Aristotle, De An. 431b25-432a3)

Aristotle says that the form of the act of knowing and the form of what is known are identical. The faculty of knowledge and the object known become isomorphic.

We can find an isomorphism like the one that Aristotle seems to have in mind in our correspondence between the norms governing discursive acts and the modality governing states. What we call "rational forms" are shared between assertions/denials and truth-makers/falsity-makers as their isomorphic roles with respect to other assertions/denials and other truth-makers/falsity-makers, respectively. Our knowledgable acts of concept use are isomorphic to the states known in these acts. The way in which these are isomorphic is that they play the same role in the opposition relations that Aristotle says are the "same manner" of opposition.

This brings us to Aristotle's idea that in order to think that things are thus-and-so, the intellect must be able to take on the form of the fact that things are thus-and-so, i.e., clause (i) in the above quote from Shields. We saw above that Aristotle says that the intellect is the form of forms because it must be able to take on any intelligible form. He considers this a respect

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matter, what thinks and what is thought are identical; for speculative knowledge and its object are identical" (Aristotle, De An. 430a4-6).

in which the intellect is rather special. In particular, Aristotle thinks that this requires that the intellect has no nature besides the capacity to take on intelligible forms.

> [S]ince everything is a possible object of thought, mind in order [...] to know, must be pure from all admixture; for the co-presence of what is alien to its nature is a hindrance and a block: it follows that it can have no nature of its own, other than that of having a certain capacity. Thus that in the soul which is called thought (by thought I mean that whereby the soul thinks and judges) is, before it thinks, not actually any real thing. (Aristotle, De An. 429a18-24)

Translated into our framework, this means that the conceptual norms that govern discourse must be such that they can match any kind of alethic-modal structure that we may encounter. The relevant notion of "encountering" an alethic-modal structure refers to that which must happen, according to Shields point (ii), when an object informs our discursive capacity. We will come back to these issues in later chapters.

What matters for our current purposes is just that the isomorphism between the normative and the alethic modal structures allows us to say that, in the ideal case, our discursive acts and the states that they represent share a common form, namely their rational form. And these rational forms are recognizably a version of the Aristotelian idea that when our intellect grasps a part of reality, it takes on the form of that part of reality. Of course, Aristotle was thinking about objects, especially substances, while we are thinking about states. Aristotle was thinking about the soul taking on the form of the object, while we think of discursive acts having the form of a state. These are important differences, but it is nevertheless helpful to see the similarities.

4.4.2 Rational Forms in Discourse and in the World

With the Aristotelian version of the general idea in mind, we can now explicate our own version of this idea in more detail.

The form of something is that in virtue of which the thing is what it is. Now, our claim is that the form of states and the forms of discursive acts are the roles they play with respect to other states and discursive acts, respectively. More specifically, we hold that they are the roles within the respective modal structures. What are these roles? For states, their roles are given by the states with which they are alethically compatible and those with which they are alethically incompatible. What it is for a state to be the state of it being cold, e.g., is to be a state that is alethically incompatible with the state of it being warm but is alethically compatible with the state of it raining, etc. If two states are alethically compatible with all and only the same states (and hence also alethically incompatible with all and only the same states), then they play the same role and they, thus, have the same form. Perhaps such states can be numerically distinct—if they occur, e.g., in different possible worlds—but they are the same kind of state.

Similarly, the role of discursive acts is given by the acts with which they are compatible and those with which they are incompatible. What it is for a discursive act to be an assertion that it is cold, e.g., is to be normatively incompatible with an assertion that it is warm but to be normatively compatible with the assertion that it is raining. If two discursive acts are normatively compatible with all and only the same discursive acts (and hence also normatively incompatible with all and only the same acts), then they play the same role and they, thus, have the same form. Such discursive acts can be numerically distinct—if they are made, e.g., by different agents—but they are the same kind of discursive act.

What we call "rational forms" are the roles of items within modal structures of compatibility and incompatibility, be they of the normative or the alethic variety. Hence, in the ideal case where our conceptual norms match the alethic-modal structure, corresponding discursive acts and states share their rational forms. An assertion of "It is cold", e.g., plays normatively the same role with respect to other discursive acts as a state that makes "It is cold" true plays alethically with respect to other states. The assertion and the state share their rational form. In general, that rational forms are shared between the two structures consists in the fact that an assertion (or denial) of ϕ is incompatible with a collection of assertions of the members of Γ and denials of the members of Δ iff any truth-maker (or falsity-maker) of ϕ is incompatible with any fusion of truth-makers for the members of Γ and falsity-makers of the members of Δ. Thus, the truth-makers of ϕ and the assertions of ϕ share their role, as do the falsity-makers of ϕ and the denials of ϕ. The assertion is isomorphic to what makes it true, and the denial to what makes it false, within their respective modal structures of compatibility and incompatibility.

We have thus uncovered the same structure in worldly states that our discursive acts represent and in these discursive acts themselves. The worldly states share their rational forms with our discursive representations of them. While we started by thinking about this structure merely on the side of discursive acts, thus following a pragmatics-first approach, we can also uncover the same structure if we start with the worldly states that our discursive acts are about. If Aristotle is right, this identity of form is necessary for our discursive acts to be about those states. But this doesn't mean that we must understand the contentfulness of our discursive acts as a matter of labeling bits of reality. We should rather strive to understand the rational form that we have found in such different kinds of matter—discursive acts and worldly states—in its own right.

4.5 Conclusion

In this chapter, we have presented a correspondence between the pragmatics-first approach that we pursued in the previous two chapters and a semantics-first approach, as it is pursued in truth-maker theory. Moreover, we have suggested that this is more them merely two isomorphic formulations of consequence relations. Rather, what this correspondence allows us to see is how discursive acts and worldly states can share their roles within their respective modal structures, i.e., their rational forms.

That discursive acts and worldly states can share their rational forms allows us to say that our discursive acts and the worldly states that they are about have something in common. And what they have in common is that in virtue of which our discursive acts have the contents they have and in virtue of which the worldly states are the kinds of states they are.

We didn't make any claim about the priority of two modal structures with respect to each other. The semantics-first approach will take the alethic-modal structure among worldly states to be prior to the pragmatic-normative structure in the order of explaining content. The pragmatics-first approach from the first two chapters will, by contrast, take the pragmatic-normative structure to be prior. While we will ultimately endorse a pragmatics-first order of explanation, the crucial question will be in what sense exactly the pragmatic-normative structure of the norms of concept use are prior.

Before we can discuss any such claim about the order of priority, we should recognize that there is a level of description—which is the level of description of this chapter—on which the two structures are on a par. All we said about their relation is that our conceptual norms ought to be such that the pragmatic-normative structure ought to match the alethic-modal structure. Our claim that the pragmatic-normative structure is prior will have to emerge against the background of this parity of the structures.

In fact, we should go one step further than recognizing the parity of the structures. We should investigate the common form of these structures in its own right. For if what we said so far is correct, then this common form is the structure of reason relations in general, not only as they appear in our discursive practice of giving and asking for reasons but also as the worldly relations about states to which our conceptual norms correspond. In order to study this abstract form of reason relations, we will generalize the semantic theory from this chapter in the next chapter. This will allow us to draw connections between this abstract form of reason relations and many familiar and interesting logics, including Linear Logic, Priest's Logic of Paradox, and Strong Kleene Logic.

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4.6 Appendix

4.6.1 Classical Logic

Definition 18. *Proof-search*: A root-first proof-search produces a proof-tree from a sequent $\Theta \succ \Sigma$, which is the root of the tree, by recursively applying the following procedure until the process terminates when the proof-tree no longer changes: (i) If $\Gamma \succ \Delta$ is the leaf of a branch of the tree at the current stage and all the sentences in Γ and Δ are atomic, then the branch remains unchanged. (ii) Otherwise, we look for the first complex sentence in $\Gamma \succ \Delta$ (starting on the left, ordering the sentences in Γ and Δ alphabetically) and build the branch up from that leaf by using the appropriate rule of CL. For example, we apply the top-to-bottom version of [L$\vee$] (moving upwards in the tree) if the left-most complex sentence in our sequent is a disjunction, etc. (Although we work with sets (and so contraction is built in), we represent the sets in our sequents with the number of copies of sentences that we get by applying this procedure to the given representation of the root, thus treating our sets (in how we represent them) like multi-sets.)

Someone might worry that this definition will yield different results for representations of the root sequent that differ in the numbers of copies of sentences. In fact, however, this doesn't happen.

Proposition 19. *Proof-searches on* $\Gamma, A \succ \Delta$ *and* $\Gamma, A, A \succ \Delta$ *yield the same results, and the same holds for proof-searches on* $\Gamma \succ A, \Delta$ *and* $\Gamma \succ A, A, \Delta$

Proof. If A is atomic, then the proof-search leaves it untouched. If A is complex, it is a conjunction, disjunction, or negation. Suppose $A = B \wedge C$. Then applying our procedure with [L$\wedge$] to $\Gamma, B \wedge C \succ \Delta$ yields $\Gamma, B, C \succ \Delta$, and applying it twice to $\Gamma, B \wedge C, B \wedge C \succ \Delta$ yields $\Gamma, B, C, B, C \succ \Delta$. Thus, the resulting set of premises is identical. The cases for [R$\vee$], [L$\neg$], and [R$\neg$] are analogous. For [R$\wedge$], applying our procedure to $\Gamma \succ B \wedge C, \Delta$ yields $\Gamma \succ B, \Delta$ and $\Gamma \succ C, \Delta$ and $\Gamma \succ B, C, \Delta$. Applying the procedure twice to $\Gamma \succ B \wedge C, B \wedge C, \Delta$ yields $\Gamma \succ B, B, \Delta$ and $\Gamma \succ C, B, \Delta$ and $\Gamma \succ B, C, B, \Delta$ and $\Gamma \succ B, C, \Delta$ and $\Gamma \succ C, C, \Delta$ and $\Gamma \succ B, C, C, \Delta$ and $\Gamma \succ B, B, C, \Delta$ and $\Gamma \succ C, B, C, \Delta$ and $\Gamma \succ B, C, B, C, \Delta$. Each of the conclusion sets in these sequents is identical to that of one of the three sequents just mentioned. The case for [L$\vee$] is analogous. ■

Proposition 20. *Proof-searches terminate, and their results are the same if we change the order of the sentences in* Γ *and* Δ.

Proof. Proof-searches terminate because the root contains finitely many logical connectives, and the children of a node always contain one fewer connective than the parent node.

To show that the order doesn't matter, it suffices to show that for each pair of rules, the order in which they are applied doesn't matter. If we have, e.g., $\neg A, B \vee C, \Gamma \succ \Delta$, applying our procedure to the first two sentences yields: $B, \Gamma \succ \Delta, A$ and $C, \Gamma \succ \Delta, A$ and $B, C, \Gamma \succ \Delta, A$. This result is the same whether we use [L¬] first and then [L∨] or the other way around. The same holds for all pairs of rules. Hence, the result of a proof-search is order-independent. ■

Proposition 21. *In* CL*, [weakening] is redundant.*

Proof. We can add the desired additional context to every application of [ID]. ■

Proposition 22. *In* CL*, the bottom-to-top operational rules can be eliminated, i.e., omitting these rules does not change which sequents are derivable.*

Proof. We argue by induction on proof-height, and look at each bottom-to-top rule in turn. Since [weakening] can be eliminated, it suffices to look at proof-trees without [weakening]. I will give the proof for [L∧]; the other cases are analogous. Suppose we have a derivation of $\Gamma, A \wedge B \succ \Delta$. We must show that $\Gamma, A, B \succ \Delta$ is derivable. If $\Gamma, A \wedge B \succ \Delta$ was derived via [L∧], we're done. For all the other rules by which $\Gamma, A \wedge B \succ \Delta$ may come, $A \wedge B$ must have been in the left context of the rule-application. We can apply our induction hypothesis and replace the conjunction with the two conjuncts. We then get $\Gamma, A, B \succ \Delta$ by applying the rule by which $\Gamma, A \wedge B \succ \Delta$ was derived in our initial proof-tree. ■

Proposition 23. *In* CL*, [cut] can be eliminated, i.e., omitting [cut] does not change which sequents are derivable.*

Proof. From the proof of Proposition 36 below, it is easy to see that CL without [cut] is equivalent to the sequent calculus of ST without the truth-rules, and it is well-known that [cut] is admissible in that sequent calculus. Hence, [cut] is admissible in CL without [cut]. ■

Proposition 24. *The sequent calculus* CL *is sound and complete with respect to classical propositional logic, i.e.* $\Gamma \mathrel{\vdash_{CL}} \Delta$ *iff* $\Gamma \mathrel{\models_{CL}} \Delta$.

Proof. For soundness, it suffices to note that every classical truth-assignment satisfies (i.e., is not a counter-model to) any instance of [ID] and that all rules of CL preserve that property, i.e., if there is a counterexample to the bottom-sequent of an application of a CL rule, then there is also a counterexample to at least one top-sequent.

For completeness, suppose that $\Gamma \succ \Delta$ cannot be derived. Hence, a proof-search for $\Gamma \succ \Delta$ yields at least one atomic sequent, $\Gamma_0 \succ \Delta_0$, such that $\Gamma_0 \cap \Delta_0 = \emptyset$. So, there is a counterexample to $\Gamma_0 \succ \Delta_0$, i.e., a classical truth-assignment that makes everything in Γ_0 true and everything in Δ_0 false. Any counterexample to $\Gamma_0 \succ \Delta_0$ is also a counterexample to $\Gamma \succ \Delta$. Since $\Gamma_0 \succ \Delta_0$ is not derivable, by the contrapositive of [cut], for any atomic sentence, p, either $\Gamma_0 \succ \Delta_0, p$ or $p, \Gamma_0 \succ \Delta_0$ is not derivable. If $\Gamma_0 \succ \Delta_0, p$ is not derivable, we make p false; otherwise $p, \Gamma_0 \succ \Delta_0$ is not derivable, and we make p true. In this way, we can extend our counterexample by assigning truth-values to all atomic sentences. Hence, $\Gamma \mathrel{\not\models_{CL}} \Delta$. ■

Proposition 25. *The rules [ID], [weakening], and [cut] are valid for* $\models_{TM}$ *iff possible states obey Exclusivity, Downward-Closure, and Exhaustivity respectively.*

Proof. Downward-Closure and [weakening]: Downward-Closure says that if $s \in S^{\Diamond}$ and $t \sqsubseteq s$, then $t \in S^{\Diamond}$. Now, if $\Gamma \mathrel{\models_{TM}} \Delta$, then any fusion of verifiers of everything in Γ and falsifiers of everything in Δ is impossible. By Downward-Closure, all states that include any such fusion as a part are also impossible. Hence, $\Theta, \Gamma \mathrel{\models_{TM}} \Delta, \Sigma$. For the other direction, suppose that [weakening] is valid and that $s \in S^{\Diamond}$ and $t \sqsubseteq s$. In accordance with

Assumption 5, let Γ, Θ, Δ, and Σ be such that s is the unique state that is a fusion of verifiers for everything in $\Gamma \cup \Theta$ and falsifiers for everything in $\Delta \cup \Sigma$. Since $s \in S^{\Diamond}$, we know that $\Theta, \Gamma \not\models_{TM} \Delta, \Sigma$. If [weakening] is valid for $\models_{TM}$, it follows that $\Gamma \not\models_{TM} \Delta$. Since $t \sqsubseteq s$, without loss of generality, let t be the state that results from s by omitting the verifiers for Θ and the falsifiers for Σ. Then t is the unique state that is a fusion of verifiers for everything in Γ and falsifiers for everything in Δ. Hence, $t \in S^{\Diamond}$.

Exclusivity and [ID]: Exclusivity says that if $s \in |p|^+$ and $t \in |p|^-$, then $\forall u(s \sqcup t \sqcup u \notin S^{\Diamond})$. So, $\Gamma, p \models_{TM} p, \Delta$. For the other direction, suppose [ID] is valid and let $s \in |p|^+$ and $t \in |p|^-$. By [ID], for any Γ and Δ, we have $\Gamma, p \models_{TM} p, \Delta$. So every state that includes a truth-maker and a falsity-maker of p is impossible, i.e., $\forall u(s \sqcup t \sqcup u \notin S^{\Diamond})$.

Exhaustivity and [cut]: Suppose that $\Gamma \not\models_{TM} \Delta$ and let u be a state witnessing this fact, i.e., a state that is a fusion of verifiers of everything in Γ and falsifiers of everything in Δ such that $u \in S^{\Diamond}$. By Exhaustivity, $\exists s \in |p|^+ (u \sqcup s \in S^{\Diamond})$ or $\exists t \in |p|^- (u \sqcup t \in S^{\Diamond})$. Therefore, either $p, \Gamma \not\models_{TM} \Delta$ or $\Gamma \not\models_{TM} \Delta, p$. But that is just what is required for the contrapositive of [cut]. For the other direction, suppose that [cut] is valid for $\models_{TM}$. Let u be possible; and, in accordance with Assumption 5, let Γ be a set such that u is the unique state that is a fusion of verifiers for everything in Γ. Hence, $\Gamma \not\models_{TM} \emptyset$. By the validity of [cut], either $p, \Gamma \not\models_{TM} \emptyset$ or $\Gamma \not\models_{TM} p$. Hence, either $\exists s \in |p|^+ (u \sqcup s \in S^{\Diamond})$ or $\exists t \in |p|^- (u \sqcup t \in S^{\Diamond})$. ■

Lemma 26. *For every top-to-bottom application of an operational rule of* CL*, the set of states deemed impossible by the bottom-sequent is the union of the states deemed impossible by the top-sequents.*

Proof. I do the case for conjunction; the proofs for negation and disjunction are analogous. For [L$\wedge$]: Note that by our semantic clauses $|A \wedge B|^+ = \{s : \exists a \in |A|^+ \exists b \in |B|^+ (s = a \sqcup b)\}$. Hence, for any Γ and Δ, we have $\{g \sqcup d \sqcup a \sqcup b : g \in |\bigwedge \Gamma|^+ \text{ and } d \in |\bigvee \Delta|^- \text{ and } a \in |A|^+ \text{ and } b \in |B|^+\} = \{g \sqcup d \sqcup s : g \in |\bigwedge \Gamma|^+ \text{ and } d \in |\bigvee \Delta|^- \text{ and } s \in |A \wedge B|^+\}$. So the states deemed impossible by $\Gamma, A, B \succ \Delta$ are identical to those deemed impossible by $\Gamma, A \wedge B \succ \Delta$.

Similarly for [R$\wedge$], note that $|A \wedge B|^- = |A|^- \cup |B|^- \cup \{s : \exists a \in |A|^- \, \exists b \in |B|^- \, (s = a \sqcup b)\}$. Therefore, $\{g \sqcup d \sqcup s : g \in |\bigwedge \Gamma|^+$ and $d \in |\bigvee \Delta|^-$ and $s \in |A \wedge B|^-\} = \{g \sqcup d \sqcup a : g \in |\bigwedge \Gamma|^+$ and $d \in |\bigvee \Delta|^-$ and $a \in |A|^-\}$ $\cup \{g \sqcup d \sqcup b : g \in |\bigwedge \Gamma|^+$ and $d \in |\bigvee \Delta|^-$ and $b \in |B|^-\} \cup \{g \sqcup d \sqcup c : g \in |\bigwedge \Gamma|^+$ and $d \in |\bigvee \Delta|^-$ and $c \in \{x : \exists a \in |A|^- \, \exists b \in |B|^- \, (x = a \sqcup b)\}\}$. Hence, the states deemed impossible by $\Gamma \succ \Delta, A \wedge B$ is the union of those deemed impossible by $\Gamma \succ \Delta, A$ and $\Gamma \succ \Delta, B$ and $\Gamma \succ \Delta, A, B$. ∎

Proposition 27. *All operational rules of* CL *are valid for* $\models_{TM}$*, i.e., if all the top-sequents are TM-valid, then so is the bottom sequent.*

Proof. Immediate from Lemma 26. ∎

Proposition 28. *If a model is a counterexample to a top-sequent of a top-to-bottom application of an operational rule of* CL*, then the model is also a counterexample to the bottom-sequent.*

Proof. By Lemma 26, if a state deemed impossible by a top-sequent is possible in $\mathcal{M}$, then a state deemed impossible by the bottom sequent is possible in $\mathcal{M}$. ∎

Proposition 29. *If we impose Exclusivity, Exhaustivity, and Downward-Closure, then* $\Gamma \models_{TM} \Delta$ *iff* $\Gamma \vdash_{CL} \Delta$ *iff* $\Gamma \models_{CL} \Delta$.

Proof. It suffices to show that $\Gamma \vdash_{CL} \Delta$ is sound and complete with respect to both consequence relations, $\models_{TM}$ and $\models_{CL}$. By Proposition 24, CL is sound and complete with respect to $\models_{CL}$. For $\models_{TM}$, we know soundness from Propositions 25 and 27. For completeness, suppose that there is no proof of $\Gamma \succ \Delta$. Hence, a proof-search for $\Gamma \succ \Delta$ yields an atomic sequent, $\Gamma_0 \succ \Delta_0$, where $\Gamma_0 \cap \Delta_0 = \varnothing$. Let $\mathcal{M}$ be a model in which $s \in S^{\Diamond}$ and $s = u \sqcup t$ such that $u \Vdash \Gamma$ and $t \dashv\!\!| \Delta$. This is a counterexample to $\Gamma_0 \succ \Delta_0$.

By Proposition 28, it follows that it is also a counterexample to $\Gamma \succ \Delta$. Since $\Gamma_0 \cap \Delta_0 = \emptyset$, such a model isn't ruled out by Exclusivity, which is the only principle that could rule out such a model. So, $\mathcal{M}$ shows that $\Gamma \not\models_{TM} \Delta$. ■

4.6.2 Relation of $\mathsf{CL}^{\setminus[\textbf{cut}]}$ to ST, LP, $\mathsf{K3}$, and TS

I will use a slightly adjusted version of the propositional fragment of Ripley's (2013) sequent calculus presentation of ST, namely the following:

Structural Rules of ST:

$$\frac{}{p \succ p}\ [\text{ID-ST}] \qquad \frac{\Gamma \succ \Delta}{\Theta, \Gamma \succ \Delta, \Sigma}\ [\text{weakening-ST}]$$

Operational Rules of ST:

$$\frac{\Gamma, A \succ \Delta \qquad \Gamma, B \succ \Delta}{\Gamma, A \vee B \succ \Delta}\ [\text{L}\vee\text{-ST}] \qquad \frac{\Gamma \succ \Delta, A, B}{\Gamma \succ \Delta, A \vee B}\ [\text{R}\vee\text{-ST}]$$

$$\frac{\Gamma \succ \Delta, A}{\Gamma, \neg A \succ \Delta}\ [\text{L}\neg\text{-ST}] \qquad \frac{\Gamma, A \succ \Delta}{\Gamma \succ \Delta, \neg A}\ [\text{R}\neg\text{-ST}]$$

$$\frac{\Gamma, A \succ \Delta}{\Gamma, Tr(\bar{A}) \succ \Delta}\ [\text{Lt-ST}] \qquad \frac{\Gamma \succ A, \Delta}{\Gamma \succ Tr(\bar{A}), \Delta}\ [\text{Rt-ST}]$$

We say that $\Gamma \models_{\mathsf{ST}} \Delta$ iff the sequent $\Gamma \succ \Delta$ is derivable in ST. Ripley (2013) uses single-line rules and an additive right-rule for disjunction, and he includes the material conditional. Given [weakening-ST], the definability of the conditional as $A \supset B =_{\text{def.}} \neg A \vee B$, and the admissibility of the bottom-to-top rules in the single-line ST calculus, these differences don't change which sequents are provable.[15] Ripley treats $\wedge$ as defined in the

[15]The proofs of these facts are straightforward, and I leave them as an exercises to the reader.

usual way, i.e., $A \wedge B =_{\text{def.}} \neg(\neg A \vee \neg B)$. Using this definition, we can show:

Proposition 30. $\Gamma \mathrel{|}\!\underline{}_{\mathsf{CL}^{+\setminus[cut]}}\, \Delta$ *iff* $\Gamma \mathrel{|}\!\underline{}_{\mathsf{ST}}\, \Delta$.

Proof. We can transform any proof-tree for $\Gamma \mathrel{|}\!\underline{}_{\mathsf{CL}^{+\setminus[cut]}}\, \Delta$ into a proof-tree for $\Gamma \mathrel{|}\!\underline{}_{\mathsf{ST}}\, \Delta$ and vice versa. Left-to-right: Applications of [ID] in $\mathsf{CL}^{+\setminus[cut]}$ can be translated into ST^{+} by [ID-ST] followed by [weakening-ST]. Applications of [weakening] are merely relabeled as applications of [weakening-ST]. Similarly for applications of the negation rules, truth rules, and [R∨] (all in both directions). Top-to-bottom applications of [L∨] are translated into applications of [L∨-ST], leaving out the third top-sequent. Bottom-to-top applications of [L∨] are translated into similar applications of [L∨-ST] with an application of [weakening-ST] if the desired sequent is the third top-sequent. Top-to-bottom applications of [L∧] are translated by putting negations of both subaltern sentences on the right by [R¬-ST], then using [R∨-ST] to get their disjunction on the right, and finally using [L¬-ST] to get the negation of the disjunction on the left. This negated disjunction is, by definition, the same as the desired conjunction. Bottom-to-top applications of [L∧] are translated by the same route in reverse. Top-to-bottom applications of [R∧] are translated by applying [L¬-ST] to the first two top-sequents, omitting the third, then disjoining the resulting negations on the left via [L∨-ST], and finally putting the negated disjunction on the right via [R¬-ST]. This negated disjunction is, by definition, the same as the desired conjunction. Bottom-to-top applications of [R∧] are translated by the same route in reverse with an addition of [weakening-ST] if the desired sequent is the third top-sequent.

Right-to-left: Applications of [ID-ST], the ST rules for negation, truth, weakening, and [R∨-ST] are translated by merely relabeling them appropriately. This leaves only [L∨-ST]. Top-to-bottom applications are translated by [weakening] to get the required additional top-sequent followed by [L∨]. Bottom-to-top applications can be merely relabeled. ∎

Proposition 31. *If we add a truth-predicate to our language, the clauses (tr+) and (tr−) to our semantics, and a sentence* $\lambda = \neg Tr(\bar{\lambda})$*, then* $\mid\!\overline{\overline{}}_{TM}$ *is trivial, i.e.,* $\forall\Gamma\forall\Delta\,(\Gamma \models_{TM} \Delta)$.

Proof. The clauses for truth imply that a state that verifies the liar sentence also falsifies it, and vice versa. For suppose $s \Vdash \lambda$. Since $\lambda = \neg Tr(\bar{\lambda})$, it follows that $s \Vdash \neg Tr(\bar{\lambda})$. By (neg+), $s \dashv\!\mid Tr(\bar{\lambda})$. And by (tr−), $s \dashv\!\mid \lambda$. The same reasoning works in reverse. Now, let s be an arbitrary verifier of λ. Hence, $s \Vdash \lambda$ and $s \dashv\!\mid \lambda$. By Exclusivity, $\forall u(s \sqcup s \sqcup u \notin S^{\Diamond})$. So, $\forall u(s \sqcup u \notin S^{\Diamond})$. Now suppose for reductio that $\Gamma \not\models_{TM} \Delta$ and let u be a fusion of verifiers for each element in Γ and falsifiers for each element in Δ. By truth-maker bilateralism, $u \in S^{\Diamond}$. By Exhaustivity, there is either a verifier or a falsifier of λ that we can fuse with u into a possible state. Without loss of generality, let that state be s. Hence, $u \sqcup s \in S^{\Diamond}$, which contradicts the earlier result. So, by reductio, $\Gamma \models_{TM} \Delta$. ■

In order to prove the completeness of $\mathsf{CL}^{+\backslash[cut]}$ with respect to $\models^{ST}_{TM}$, we cannot use the technique of proof-searches from above because proof-searches are no longer guaranteed to terminate. Hence, I follow Ripley (2013, 162-63) in using the technique of (possibly infinite) reduction trees from Takeuti (1987).

Definition 32. *Reduction tree*: The reduction tree for a sequent, $\Gamma \succ \Delta$, is the possibly infinite tree that results from starting with $\Gamma \succ \Delta$ as the root of the tree and then extending at each stage each top-most sequent of the tree as follows, until all branches are closed or else extending the tree ω-many times: (i) If the sequent is an axiom, i.e., is such that the left and the right side share an atomic sentence, then the branch remains unchanged and is closed. (ii) If the sequent has the form $\Gamma, \neg A \succ \Delta$ or $\Gamma \succ \neg A, \Delta$ and no reduction has been applied to $\neg A$ in previous stages, they reduce to $\Gamma, \neg A \succ A, \Delta$ and $\Gamma, A \succ \neg A, \Delta$ respectively. (iii) If the sequent has the form $\Gamma, A \wedge B \succ \Delta$ and no reduction has been applied to $A \wedge B$ in previous stages, it reduces to $\Gamma, A, B, A \wedge B \succ \Delta$; and if it has the form $\Gamma \succ A \wedge B, \Delta$ and no reduction has been applied to $A \wedge B$ in previous stages, the reduction tree branches into $\Gamma \succ A \wedge B, A, \Delta$ and $\Gamma \succ A \wedge B, B, \Delta$ and $\Gamma \succ A \wedge B, A, B, \Delta$. (iv) Similarly, $\Gamma \succ A \vee B, \Delta$ reduces to $\Gamma \succ A \vee B, A, B, \Delta$; and $\Gamma, A \vee B \succ \Delta$ reduces to $\Gamma, A, A \vee B \succ \Delta$ and $\Gamma, B, A \vee B \succ \Delta$ and $\Gamma, A, B, A \vee B \succ \Delta$. (v) $\Gamma, Tr(\bar{A}) \succ \Delta$ reduces to $\Gamma, A, Tr(\bar{A}) \succ \Delta$; and $\Gamma \succ Tr(\bar{A}), \Delta$ reduces to $\Gamma \succ Tr(\bar{A}), A, \Delta$.

Lemma 33. *The set of states deemed impossible by a sequent is the union of the states deemed impossible by the sequents to which it reduces in a reduction tree.*

Proof. We look at each clause in the reduction procedure. The lemma holds trivially for clause (i). It holds for (ii) because the truth-makers of $\neg A$ are exactly the falsity-makers of A, and vice versa. The other cases, in particular those for (v), are analogous except for when the reduction tree branches out, such as in the case of $\Gamma \succ A \wedge B, \Delta$. In this case, the lemma holds because the falsity-makers of $A \wedge B$ are the union of the falsity-makers of A, the falsity-makers of B, and any fusion of such falsity-makers, which corresponds to the three sequents that result from the reduction. ■

Definition 34. *Sequents resulting from an open branch of a reduction tree*: If an open branch of a reduction tree terminates, the resulting sequent is the leaf of that branch. If the open branch does not terminate, then the resulting sequent is the sequent $\Gamma_\omega \succ \Delta_\omega$, where Γ_ω is the union of all the sets on the left side of sequents in this open branch and and Δ_ω is the union of the sets on the right side of sequents in the branch.

Lemma 35. *Let $\Gamma \succ \Delta$ be a sequent resulting from an open branch, let Γ^{at} be the set of atomic sentences in Γ, and let Δ^{at} be the set of atomic sentences in Δ. Then a state that is deemed impossible by $\Gamma^{at} \succ \Delta^{at}$ includes as a part a truth-maker for every sentence in Γ and a falsity-maker for every sentence in Δ.*

Proof. We argue by induction on the complexity of sentences in $\Gamma \cup \Delta$. The states deemed impossible by $\Gamma^{at} \succ \Delta^{at}$ trivially include truth-makers for every atomic sentence in Γ and falsity-makers for every atomic sentence in Δ. Suppose our lemma holds for sentence up to complexity n, and let's consider sentences of complexity $n+1$. Note that since we have an open branch, we know that all possible reduction procedures have been applied. For negations in Γ, we know that the negatum, which is of complexity n, is in Δ. So, by our induction hypothesis states deemed impossible by $\Gamma^{at} \succ \Delta^{at}$ include a falsity-maker for the negatum, which is a truth-maker for our negation. Similarly for all other connectives where the reduction tree does not branch. For disjunctions on the left, we know that Γ contains also one or both of the disjunctions, which are of complexity n. So by our hypothesis, $\Gamma^{at} \succ \Delta^{at}$ contains truth-makers for one or both disjuncts, and any of these options ensures that it includes a truth-maker for the disjunction. Similarly for conjunctions on the right. ■

Proposition 36. $\Gamma \vdash_{\mathsf{CL}^{+\setminus[cut]}} \Delta$ *iff* $\Gamma \models^{ST}_{TM} \Delta$.

Proof. Left-to-right: We leave out [cut] and Exhaustivity in Propositions 25, and the proof still shows the validity of [ID] and [weakening]. Since the operational rules haven't changed, the validity proof for the operational rules from Proposition 27 still applies. Clauses (tr+) and (tr−) ensure that [Lt] and [Rt] are valid for $\models^{ST}_{TM}$.

Right-to-left: Suppose that there is no proof of $\Gamma \succ \Delta$. Hence, a reduction tree for $\Gamma \succ \Delta$ has an open branch. Let $\Gamma_\omega \succ \Delta_\omega$ be the sequent that results from that branch, and let $\Gamma^{at}_\omega \succ \Delta^{at}_\omega$ be the sequent that results from $\Gamma_\omega \succ \Delta_\omega$ by omitting all complex sentences. We can use as our desired counter-model any model that makes possible one of the states deemed impossible by $\Gamma^{at}_\omega \succ \Delta^{at}_\omega$. For by Lemma 35, any state that is deemed impossible by $\Gamma^{at}_\omega \succ \Delta^{at}_\omega$ is also deemed impossible by $\Gamma_\omega \succ \Delta_\omega$. And by Lemma 33, any state deemed impossible by $\Gamma_\omega \succ \Delta_\omega$ is deemed impossible by $\Gamma \succ \Delta$. So our model is a counter-model to $\Gamma \succ \Delta$. We know that there is such a model because any model will work that makes only those states impossible that are required to be impossible by Exclusivity, and $\Gamma^{at}_\omega \cap \Delta^{at}_\omega = \emptyset$. ■

Note that the models that make possible all or some the states deemed impossible by $\Gamma^{at}_{\omega} \succ \Delta^{at}_{\omega}$ can respect our semantic clauses for the transparent truth-predicate, even though the atomic sentences in $\Gamma^{at}_{\omega} \succ \Delta^{at}_{\omega}$ contain truth-predicates to complex sentences. For if, e.g., $Tr(\overline{A \wedge B}) \in \Gamma^{at}_{\omega}$, then $A \wedge B \in \Gamma_{\omega}$. So the atomic part of the reduction of $A \wedge B$ is also in Γ^{at}_{ω}. And if A or B include truth-predications, this will apply to those again.

4.6.3 Non-Monotonic Consequence

Proposition 37. $\Gamma \mathrel{\vert\!\widetilde{\sim}^{\mathfrak{B}}_{NM}} \Delta$ *iff* $\Gamma \mathrel{\models^{NM_{\mathfrak{B}}}_{TM}} \Delta$.

Proof. Left-to-right: By induction on proof-height. The axioms of $\vert\!\widetilde{\sim}^{\mathfrak{B}}_{NM}$ are all the sequents in $\mathfrak{B}$. Hence, they are valid according to $\models^{NM_{\mathfrak{B}}}_{TM}$. Our sequents rules preserve validity in TM-models by Proposition 27.

Right-to-left: Suppose that $\Gamma \succ \Delta$ is not derivable in $\mathsf{NM}_{\mathfrak{B}}$. A root-first proof-search must yield an atomic sequent that is not in $\mathfrak{B}$. Hence, the states that this atomic sequent deems impossible are not all impossible in all models. So we can find a model in which one of these states is possible. By Proposition 28, this is also a counterexample to $\Gamma \mathrel{\models^{NM_{\mathfrak{B}}}_{TM}} \Delta$. ■